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Learning as a critical encounter with the Other: Prospective teachers conversing with the history of mathematics

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Abstract
In this paper, drawing on the philosophy of dialectical materialism, we present an elaboration of two concepts that tend to remain backstage in debates in the field of history in mathematics education, namely, the concept of *mathematics learning* and the concept of classroom *mathematics knowledge*. The elaboration of these two concepts makes room to return to the longstanding question of the role of history in mathematics education. We argue that the history of mathematics in education is not a choice but a need—a central part of the process of understanding our human nature as essentially historical and cultural. We illustrate these ideas through the analysis of the encounter of a group of prospective teachers with a 14th century problem about motion. In this encounter the students engaged critically in conversations with voices of the past, while restoring the aesthetic, ethical, and historical dimensions of knowing and learning.

Keywords. Objectification • subjectification • otherness • ethics • dialectical materialism

1. Introduction

Our study is embedded in a well-established educational research field located at the intersection of the history of mathematics and the pedagogy of mathematics. This intersection has given rise to an interdisciplinary collaboration between mathematicians, mathematics educators, historians, epistemologists, and teachers of mathematics.[1] Certainly, important theoretical as well as practical results have been obtained in recent decades (for some seminal work see, for example, Barbin, 1997; Clark, 2019; Fauvel & Maanen, 2000; Furinghetti, 1997; Jahnke, 2014; Jankvist, 2009). However, as Barbin, Guillemette and Tzanakis (2020) noted in their recent entry to the *Encyclopedia of Mathematics Education*, there is still an urgent need to close the gap between theoretical and empirical research. In highlighting some points requiring more research, these authors mention the need to investigate in further detail "the historicity of mathematical conceptions and practices," and make a call for developing new "specific themes of research" (p.

[1] See, for example, the *International Study Group on the Relations between History and Pedagogy of Mathematics* (HPM) (http://www.clab.edc.uoc.gr/HPM/) and the *European Summer University on the History and Epistemology in Mathematics Education*.

340). In our study we respond to this call in a twofold way. First, by presenting some results from a study that is both theoretical and empirical, and second, by focusing on two themes that are central in the intersecting sites of the history of mathematics and the pedagogy of mathematics—two themes that, to an important extent, have remained backstage in the discussions. The first theme is the concept of *mathematics learning*; the second theme is the concept of classroom *mathematics knowledge* (i.e., the mathematics knowledge that the students encounter in the school). In this paper we offer a theorization of these themes from the philosophical viewpoint of dialectical materialism.

We begin with an attempt to reconceptualize classroom mathematics knowledge. To do so, we suggest that mathematics knowledge can be conceived of as a *dynamic* and fluid *system*, that is, a system of ways of mathematically thinking, reflecting, and doing that have been historically and culturally constituted. The historical investigation of those ways of mathematically thinking, reflecting, and doing requires attending to what mathematicians were doing in their specific historical, social, cultural, and political times. In this view, mathematics knowledge is not a psychological or subjective entity; it is a cultural-historical one that unavoidably embeds and conveys the conflicting societal views and antagonistic forces of its own production. It embodies, in sublated or sedimented ways, the struggles of its historical refinements.

Against this backdrop, we suggest that mathematics *learning* can be theorized as a *critical encounter* with mathematics knowledge. More specifically, conceiving of this encounter in a dialectical-materialist sense, the students' mathematics learning is considered a process in which knowledge (necessarily *historical* knowledge, for how could knowledge be otherwise?) comes to life and, through its sensible appearance, is disclosed to the students' consciousness through dynamic endeavors of meaning-making. In this encounter, teachers and students enter into sensible, material, semiotic, embodied, and dialogical relationships with knowledge, and thereby with other voices, past and present. The dialogism we have in mind here has profound implications not only for understanding mathematics, but also for those who are understanding it. Indeed, in the appearing of knowledge, teachers and students are led to position, reposition, and co-position themselves vis-à-vis knowledge and the classroom practice of mathematics. The result is that learning is not just about *knowing*, but also about *becoming*—that is, about the constitution of teachers and students as cultural and historical subjects. These ideas lead us to argue that the history of mathematics in education is not a choice but a need—a central part of the process of understanding our human nature as essentially historical and cultural.

The paper is organized as follows. In the next section we articulate in some detail the aforementioned theoretical views of knowledge and learning. This section, theoretical in nature, is followed by sections in which we discuss a classroom activity in which prospective teachers engaged in understanding a historical mathematical problem about motion—a problem found in ~~Piero~~ Paolo dell'Abbaco's 14th century *Trattato d'Aritmetica*. Before discussing the way the students engaged in the historical problem and their conversation with voices of the past (Boero, Pedemonte, Robotti, & Chiappini, 1998), we discuss dell'Abbaco's problem, highlighting the inarticulable articulation of time before time became a measurable object. In paying attention to the students' dealings with dell'Abbaco's problem, we attempt to trace the dissonances and resonances, and tensions and conflicts that emerged in the unfolding awe of the encounter with the historical Other. We argue that, in this encounter, the students created a space that is neither theirs nor dell'Abbaco's; they created a *third* space, an in-between space, the space of the reception of historical mathematics knowledge. We call this the *joint space of past and present presence*, a space where past and present are continuously re-writing each other, dialectically.

2 Mathematics knowledge and learning

2.1 Knowledge as a general cultural-historical entity

In *The Philosophy of History*, Hegel (2001) invites us to conceptualize knowledge along the lines of a dynamic organic system that is made of sub-systems comprising ideational objects, which he calls *ideas*. These ideas (e.g., the idea of tangent or number) are neither subjective nor mental entities. Their main characteristic is to be "general and abstract" (Hegel, 2001, p. 36). Drawing from Kant, Hegel says that they "exist for themselves"; that is, they exist independently of any empirical individual. But, contrary to Kant and his enduring Platonism, he argues that ideas are not transcendental entities (*things-in-themselves*). In Hegel's view, there is a profound relationship between the conceptual and the material world, as attested to by the manner in which ideas are *in* things.

First, ideas offer individuals possibilities for thinking and doing. In this sense ideas are *possibility*, *potentiality.* Were we born in Plato's time, we would have found ourselves in a world with different potentialities to think about law, school, and mathematics, for example.

Second, ideas come into life, that is, into concrete existence. Through their concretion, they become embodied *in* things, *in* actions, *in* language, *in* symbols.

2.2 The ascent of knowledge from the abstract to the concrete

Now, how do ideas come into concrete existence? Hegel's answer is, through activity, "the activity of man [sic] in the widest sense" (Hegel, 2001, p. 36). The idea of a tangent, for example, is a *general* (it is *potentiality)*. What brings it to life is human activity. By being brought to life, the tangent comes to be *in* things, and can become an object of thought and analysis.

A case in point is the remark made by the mathematician and didactician Laurent Vivier (2020) in a recent seminar in Paris. Vivier remarked that we are unable to deal with the notion of tangent in general. A tangent *as such* is not thinkable. A tangent cannot show itself. To become an object of thought, to come into existence, we need a context—for example, a functional, cinematic, or geometrical context, in which to do things; in short, we need an activity. Through its appearing in human sensuous-intellectual-material-contextual activity, ideas or knowledge can be generalized, expanded, or transformed, and new ideas can emerge.

The appearing of knowledge through activity is the process that in dialectical materialism is called *the ascent from the abstract to the concrete*. In this process, knowledge becomes "the unity of subject and object, of form and content" (Knox, cited in Hegel, 2008, p. xxv).

Hegel's work has the merit of stressing the importance of *human activity* in the production of knowledge. It allows us to envision mathematics as a *practice* where individuals produce and reproduce knowledge and in doing so co-produce themselves.[2] Now, for Hegel, the knowledge

[2] The conceptual shift from mathematics as a formal system to a system of practices in historical and philosophical trends can be found in Mancosu's (2008) work. See also Otte (1994).

produced in a certain historical time and a certain cultural formation is the manifestation of an abstract *Spirit* or *Mind* (Hegel, 1977). Marx (1998) argued that the relation between history and knowledge must be understood the other way around. What is being manifested in every cultural-historical activity is not an abstract *Spirit*, but the ideational counterpart of concrete human activity. So, while Hegel explained practice, activity, labor, from the Mind, from the Idea, Marx claimed that we need to explain Mind and "the formation of ideas [and knowledge] from material practice" (Marx, 1998, p. 61).[3]

2.3 Learning

An articulation of these dialectical materialist ideas in the field of education is what, in part, is attempted in the theory of objectification (Radford, 2021). Knowledge is considered a continuously evolving, complex system of *ways of thinking, reflecting,* and *action* that was already there before each one of us was born. It is through the activities in which we engage that we encounter knowledge.

In this line of thought, *learning* is a critical encounter with knowledge. Before learning happens, before encountering knowledge, knowledge is *potentiality*, a historical and cultural generative capacity for action and thought—for example, the mathematician's contemporary mathematical forms of action and reflection. In this encounter, we are faced with the *alien*, the *Other*. The encounter with knowledge is the primal acknowledgement of Otherness and its fundamental role in our experience of the world. This is why the encountering of knowledge (learning) is the mark of a *difference* between self and something else (the Other). When we learn we feel this encounter as the encounter with something that *objects* us—etymologically speaking, something that is set against or that opposes us. It is indeed from the Latin terms "*ob-jacere*," "*objectare*," that the theory borrows its name (Radford, 2021, p. 77).

In the context of the school, the students' encounter with knowledge is underpinned by mathematics classroom *activity*. It is through classroom activity that knowledge is brought to life and finds itself embodied in the procedures, discussions, and all semiotic activity that the teacher and the students carry out to pose, solve, discuss, and reflect on problems.

2.4 Processes of objectification and subjectification

In order to theorize mathematics learning, we distinguish two processes in classroom activity. One is termed processes of objectification. The other is termed processes of subjectification. Processes of objectification are the social processes of progressively becoming critically conscious of cultural-historical systems of thinking and doing—something that students gradually notice and at the same time endow with meaning (Radford, 2021). Processes of subjectification are based on the idea that we, humans, are always unfinished projects of life, subjects perpetually in the making. Processes of subjectification are the processes of the continuous creation and co-creation of a singular (and unique) historical and cultural subject. They are defined as those processes whereby

[3] An encompassing discussion of Marx's critique of Hegel can be found in Fischbach's interesting translation and commentary of Marx's 1844 *Philosophical and Economic Manuscripts* (Marx, 2007).

teachers and students position themselves, while at the same time are positioned by others against the always contested backdrop of culture and history.[4]

Against this theoretical background, the mathematics educational problem is to offer the students opportunities, occasions to encounter mathematics knowledge in rich and meaningful ways—more specifically, to offer the students opportunities to enter into conversation with culture by critically co-positioning themselves vis-à-vis mathematics knowledge in/through its practice.

2.5 Critically co-positioning

Co-positioning refers to the students' agentic movement in the classroom activity that makes knowledge appear. It refers to the way the students *assert* themselves as mathematical subjectivities in the practice of mathematics. Now, the educational goal, we contend, is not just to create the conditions for the students to encounter knowledge (this is what direct teaching and reproductionist pedagogies do). To move away from disempowering pedagogies, a *critical* stance vis-à-vis the knowledge that is being encountered is required. Hence the critically co-positioning gerund in the sentence above. This critical stance entails a sensibility that makes us appreciate that the knowledge that is being encountered offers us *a* possible way (as opposed to *the* way) to interpret and to think about the world. The sensibility at the base of the critical stance makes us also realize that the knowledge that is being encountered conveys *certain* views, voices, and assumptions, and carries with it its own limits and possibilities.

To investigate the key idea of co-positionality, we resort to the construct of 'voice'. Finding one's voice or having a voice is "moving from silence into speech," "a gesture of defiance that heals, that makes new life and new growth possible" (bell hooks, 2015, p. 29), something that "assumes a primacy in talk, discourse, writing, and action" (p. 33).[5] Voice, hence, involves *more* than language and discursive activity. We argue that in the embodiment of knowledge—in its movement from potentiality to actuality, in the ascent from the abstract to the concrete—there is always a surplus that escapes language, something that was going to be said but fails to enter the realm of language. This is the difference that Emmanuel Lévinas (1974) makes between "the saying *(le dire*)" and "the said *(le dit)*." The failing of the said in capturing knowledge through language does not derive from a technical linguistic difficulty. The failing is part of the ontological nature of knowledge and its embodiment; the failing of language in grasping and seizing knowledge only shows its epistemological limits (Radford, 2003). Knowledge as a system of thinking and doing escapes each one of its possible determinations. Yet, it is only through a *determined* form—that is, the concrete and contextual form it acquires through human activity in coming into life, in becoming an object of discourse, perception, symbolization, and tactile action—that those systems of thinking and doing can be apprehended and encountered. This is why in this view mathematics is both ideational and material. Mathematics is visual, tactile, aural, material, artifactual, gestural, and kinesthetic—something produced by the joint labor of the teachers and the students.

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[4] In fact, processes of objectification and subjectification go hand in hand, but we distinguish them for analytical purposes.

[5] Voice is an irruption of self into the social plane. In voicing something, the student dares to show herself, to expose herself, and to become part of the classroom practice. At the same time, in voicing something, the classroom practice is altered. This irruption/exposition/alteration is what the term 'defiance' tries to capture.

2.6 The role of the history of mathematics

What is the role of the history of mathematics in this approach? For one thing, the history of mathematics is not a tool to improve the students' learning of mathematical contents. Nor is it meant to expand the students' culture. In the conception of history as a tool, the history of mathematics, like any tool, is seen as something *external* to its object, just as the screwdriver is external to the screw. In the view we are articulating here, the history of mathematics is a *necessary* part of the conversations that underpin learning. And for this same reason, the history of mathematics cannot be seen as a mere humanistic expansion of the students' culture. The history of mathematics is part and parcel of education—that is, of the students' and teachers' encounter with other voices and other cultures. It is part of the crucial process of the making of the students, of their unending process of making themselves a meaningful place in the world.

The necessity that we are claiming for the inclusion of the history of mathematics in mathematics education needs to be understood in light of our conceptions of knowledge, learning, the student, and the teacher. Teachers and students are not conceived of as entities that generate each one from *within,* which is the view that constructivist approaches adopt in claiming that it is the student who constructs *her or his* own knowledge and experiential reality (von Glasersfeld, 1995), even if they do so in interaction with others (Cobb, 1998). On the contrary, in our account, teachers and students find the very fabric of their being and existence in the threads of culture and history and the social world. Conceiving, hence, of teachers and students as beings whose nature is crafted out of the social world in a *definite* way, and learning as a critical positioning vis-à-vis cultural knowledge, makes the history of mathematics in mathematics education a necessity. Indeed, the history of mathematics becomes a necessary component of the vital educational project of understanding our human nature as essentially *historical* and *cultural*, and understanding that even our most creative deeds are only possible as we draw on cultural-historical systems of thinking. Just as we did not invent the languages we came to speak, we do not invent (or reinvent) mathematics. We *encounter* mathematics, which does not preclude us from making novel contributions to it. In encountering mathematics, we engage in it, we enjoy it, we take a critical position towards it, and we can expand and transform it.

In what follows we present some episodes of the encounter of a group of prospective teachers with the mathematical representation of motion as it appears in a Renaissance text. Before addressing the students' conversation with history, we comment on some aspects of the Renaissance mathematical approach to motion that didactically guided the encounter we offered to the students.

3. Motion in the Renaissance

Although they were not as prominent as the mercantile problems found in manuscripts of the abaci teachers, mathematical problems about motion gained some popularity in the Renaissance. Some present a similar scenario to a problem found in a school textbook, *Problems to Sharpen the Young*, by Alcuin of York from the so-called 8th century Carolingian Renaissance.

3.1 Alcuin's problem

Alcuin's problem reads as follows:

> There is a field 150 feet long. At one end is a dog, and at the other a hare. The dog chases when the hare runs. The dog travels 9 feet in a jump, while the hare travels 7 feet. How many feet will be travelled by the pursuing dog and the fleeing hare before the hare is seized? (Alcuin, 2005, p. 68)

The problem lets us get a glimpse of the manner in which, at this point in the Middle Ages, speed and time became objects of mathematical inquiry. Space is not only mentioned in the problem but also *measured* by what Alcuin refers to as "feet". By contrast, time is not explicitly mentioned. The question asked in Alcuin's problem is not "*how long* will it take for the dog to catch the hare?" The question is about space: how many feet will the dog travel before seizing the hare. Alcuin provides his addressee with the distance separating the dog and the hare and their "speeds". The speeds are expressed in terms of the idea of *jump*. In a jump, the dog travels 9 feet, *while* the hare travels 7 feet.

How then, without employing the modern idea of time, can this problem be solved?

Let us turn to the solution. Alcuin says:

> The length of the field is 150 feet. Take half of 150, which is 75. The dog goes 9 feet in a jump. 75 times 9 is 675; this is the number of feet the pursuing dog runs before he seizes the hare in his grasping teeth. Because in a jump the hare goes 7 feet, multiply 75 by 7, obtaining 525. This is the number of feet the fleeing hare travels before it is caught. (Alcuin, 2005, p. 68)

Alcuin takes a foot, the measuring unit of space, as something abstract: it is not the hare's foot, nor the dog's. To somehow mark the unfolding motion of the moving bodies (the dog and the hare), Alcuin resorts to the idea of jump (*saltu* in the original Latin). It evokes a phenomenological movement that *unfolds over a certain duration.* After each jump, the dog comes 2 feet closer to the hare. The first calculation (that is, the half of 150 feet) corresponds to the number of jumps that will occur before the dog seizes the hare. Time thus appears in the problem only in this oblique way. Consequently, the dog will need 75 jumps to catch the hare. This number of jumps is multiplied by the 9 feet that the dog goes in a jump and then by 7; that is, the number of feet that the hare goes in a jump. The resulting numbers are the feet traveled by each animal.

3.2 ~~Piero~~ Paolo dell'Abbaco's problem

Problems like Alcuin's became popular later on. There is a problem in a 14th century Italian manuscript composed by ~~Piero~~ Paolo dell'Abbaco that is one of the problems that we used in the activity with our prospective students. The problem reads as follows:

> A fox is 40 paces ahead of a dog, and 3 paces of the latter are 5 paces of the former. I ask in how many paces the dog will catch the fox. (dell'Abbaco, in Arrighi, 1964, p. 78)

As in Alcuin's problem, time is mentioned implicitly through motion. However, in dell'Abbaco's problem, there is no such idea as jump. To express time, dell'Abbaco uses the term "paces" (*paxxj*) in two senses. First, pace is used to *measure space* (as Alcuin used "feet" in the problem discussed above). This sense is the one appearing in the statement of the problem: there are 40 paces separating the fox and the dog. Second, pace is used in the sense of a counter of the reappearing events (a kind of pendulum, so to speak). This second sense of pace is not clear in the statement of the problem. It is only revealed in the solution of the problem. Let us look at dell'Abbaco's solution:

Do in this way: if 3 is worth 5, how much is 5 worth? Multiply 5 by 5, which is 25, and divide by 3, you will have 8 1/3. Now you may say: for each 5 of those [paces] of the dog, you have 8 1/3 [paces] of the fox; so the dog approaches the fox 3 1/3 [fox's paces]. In how many paces will he [the dog] reach her [the fox] by [covering] 40 paces? Then say: if 5 are worth 3 1/3, for 40, how many will I have? Multiply 5 by 40, which is 200, and divide by 3 1/3. Bring [i.e., reduce] to thirds, thus multiply 3 by 200, which makes 600, and divide by 3 1/3, that is 10/3 and then divide 600 in 10, it gives 60. And the dog will do 60 paces before it reaches the fox. And it is done. And the proof is that in 60 paces the fox goes 60, and the dog in 60 paces is worth 100 [i.e., 60 paces of the dog are worth 100 paces of the fox], because three of his [dog's paces] are worth 5 [of the fox]; therefore 60 paces [of the dog] are worth a good 100 [of the fox]. It is done." (Arrighi, 1964, p. 78; our translation)

In the first part, dell'Abbaco draws on the original datum: 3 dog's paces are equal to 5 fox's paces (if D stands for the dog's pace and F stands for the fox's pace, we would have $3D = 5F$). He then calculates 5D. He determines that $5D = 8\frac{1}{3} F$. Now, we need to see "pace" in the second sense; dell'Abbaco assumes that while the dog makes 5 paces, the fox makes 5 paces as well (we have here pace as a counting marker of the dog's and the fox's motions). Switching now to the spatial sense of pace, he can now assert that the fox has traveled 5 fox-paces and deduce that, in 5 paces, the distance between the fox and the dog diminishes by 3 1/3 fox-paces. Knowing that they are 40 fox-paces apart, and continuing to use the rule of three, dell'Abbaco concludes that the dog will need to go 60 dog-paces to reach the fox. In the proof that he adds at the end of the problem, we see the second sense of "pace" in a clear way: "And the proof is that in 60 paces [60 ticks of the pendulum] the fox goes 60" [fox-paces] (pace as distance).

The implicit nature of time in dell'Abaco's problem can be better understood if we bear in mind that issues like the elucidation of time, determination of its nature, and forms for calculating it were not pressing in the sociocultural context of medieval activities. People organized their lives around the cycle of the seasons. Church time marked the canonical hours: *prime* was around the beginning of the day or first hour, *terce* was the middle of the morning, *nones* was midday, *vespers* corresponded to the middle of the afternoon, and *compline* meant the end of the day. The unit of labor time was the day, defined by sunrise and sunset. Time was hence something mediated by aural and visual experiences (Radford, 2009). The emergence of modern science, Heidegger suggests, is related to "the question of measurability," something requiring "the representation of a thing as an object in its objectivity, which is the possibility for measuring it." Measuring entails the transformation of "presence [*Anwesenheit*]—mere estimation—[into] the foundation of quantitative measuring," that is to say, "the manner in which the human being measures himself with things" (Heidegger, 2001, p. xxvi).

4. Conversing with the Renaissance mathematics of motion in pandemic times

In this section we discuss a two-session teaching-learning activity involving prospective teachers of an Italian Faculty of Education. The teaching-learning activity involved 14 students of a MA program in education, and one of the authors of this article (G.S.) who teaches mathematics education in the program. Once graduated, the students receive a national qualification to teach in Italian primary schools. The Faculty of Education curriculum provides two courses in mathematics and mathematics education and two laboratories in mathematics education. At the end of their studies, the students have been exposed to basic mathematics content and some major themes in mathematics education. Most of the students come from language or human sciences high schools where mathematics is not one of the core subjects in the curriculum.

The teaching-learning activity reported here was part of the Laboratory of Mathematics Education. Due to the COVID-19 pandemic the teaching-learning activity did not happen in person; it was carried out online via the Teams platform. The students were divided into six groups; each student had their microphones and video cameras turned on. The activity and the interaction between the students and the teacher were carried out according to the principles of *joint labor* (Radford, 2021) where teacher and students work in concert. We chose the group of Maria and Anna (pseudonyms), as this group was the best paradigmatic match to the observed virtual laboratory phenomena. Maria and Anna were working at the same desk with the computer's camera recording both of them.

The online setting imposed some limitations on the data that can usually be recorded in face-to-face classrooms. The video recording provided by Teams sometimes left the participants' kinaesthetic activity out of the recording frame. Nevertheless, the video recordings were sufficient to carry out our semiotic analysis and pinpoint significant instances of gesturing and kinesthetic activity (for information about the semiotic multimodal analysis, see, e.g., Radford, 2015). Some sketches from the recording presented below include partial reconstructions.

We invited the prospective teachers to engage in two historical problems. Due to space constraints, we discuss here the results from only one problem—the dell'Abbaco problem mentioned above, which was the focus of the first session of the teaching-learning activity.

To engage in the problem, the students were invited to read dell'Abbaco's problem both in Vulgar and Modern Italian.[6] Then, they were invited to solve the problem arithmetically and algebraically. After that, the students read dell'Abbaco's 14th century solution of the problem. They were encouraged to talk and express their thoughts about dell'Abbaco's solution. In particular, the students were encouraged to reflect on how time and space were conceived both in the formulation and the solution of the problem. The students were also encouraged to formulate a similar problem in a modern context (without foxes and dogs) and identify the link with the historical problem. Finally, the students were asked to solve their problem using dell'Abbaco's method. We focus on the arithmetic solution that the students offered and on their reaction to dell'Abbaco's solution.

4.1 Materializing movement

The students started by reading the statement of dell'Abbaco's problem and delved immediately into a discursive and abstract approach, looking for the correct calculations to find the number of paces the dog has to run in order to reach the fox.

1. Maria: It's 40 paces (*She gestures confused and embarrassed; she starts laughing*).
2. Anna: 3 paces of the dog are equivalent to 5 of the fox . . . mmh . . .
3. Maria: The 40 paces are of the fox . . . (*She is not convinced*). 40 paces of the fox are 120 of the dog. If one pace of the dog . . . (*The students laugh*). So, excuse me, 40:5 = 8 (*in*

[6] In the original:

> Una volpe è innanzj a un chane quaranta paxxj, e ongnj tre paxxj di queglj del chane sono 5 di queglj della volpe.Vo' sapere in quantj paxxj la giungnerà. (dell'Abbaco, in Arrighi, 1964, p. 78)

synchrony with Anna) 8 x 3 = 24. (*Anna lowers her voice saying "as an absolute value" and they both laugh, embarrassed and puzzled*).

4. Anna and Maria: Let's write it, you never know (*laughing bewilderedly).*

As we can see, after reading the text of the problem, the prospective teachers laughed in embarrassment as if they had no idea how to proceed. Although the students were acquainted with arithmetic problems and the basic notions of mechanics learned in high school and in physics courses they attended at the Faculty of Education, the manner in which dell'Abbaco's problem was formulated was felt as something alien. It is this sense of estrangement that produces a historical text, as Barbin (1997) and Guillemette (2017) have pointed out in their work. The historical problem brings with it its own world, its world of concepts and meanings; it brings with it a world of thinking and talking about things that is *different* from our world of thinking and speaking. Laughing is the way students acknowledge this difference and is also a form of positioning themselves as subjectivities vis-à-vis dell'Abbaco's voice. It is recognizing the tremendous tension between diachronic and synchronic reason—the tension between past and present.

5. Maria: She runs 40 paces. A pace of the fox . . . In 5 paces of the fox the dog makes 3 paces. 5 paces of the fox are 3 of the dog. So to cover 40 paces . . . 40 paces of the fox are 8 . . . 8 slots of the dog, of advancement of the dog.
6. Anna: Uh uh, yes (*puzzled*).
7. Anna and Maria: Each slot is 3 paces . . .
8. Maria: Excuse me, 3 x 8 is 24 (*The students laugh*).
9. Anna: I am turning crazy!
10. Maria: (*astonished and puzzled)* But the fox continues jumping forward?!
11. Teacher: (*moving two pens, simulating the motion of the fox and the dog; see Figure 1.1*)[7] . . . we want to know after how many paces the dog reaches the fox.
12. Maria: (*replicating with gestures the dog chasing the fox as if the fox were still, see Figure 1.2 and 1.3).* If the fox were still the dog would reach it in 24 paces. But the fox does not stay still!

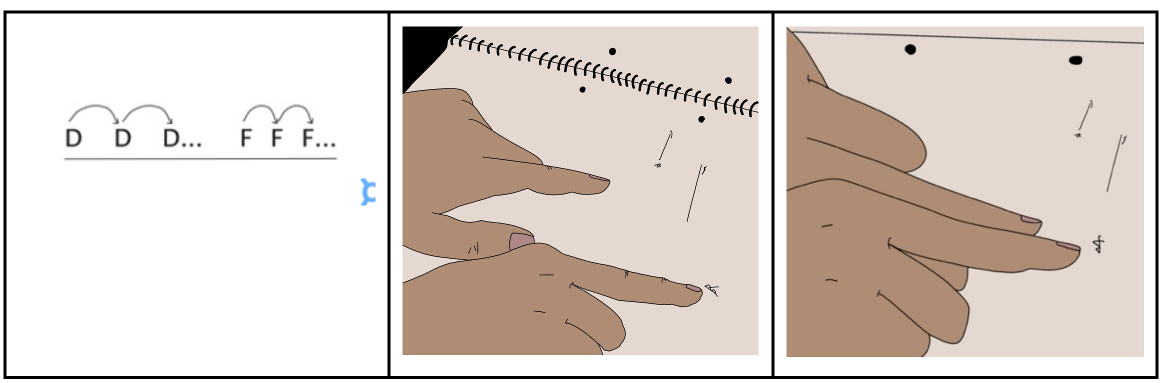

Fig. 1. In Figure 1.1, the teacher embodies with his fingers the simultaneous movement of the two animals as the dog chases the fox. In Figure 1.2 and 1.3, Maria embodies her enactment of dell'Abbaco's problem in which the fox does not move.

The students' first response to dell'Abbaco is circumscribed by the mathematical conceptualizations that appear in the statement of the problem. It is as if, in tune with the text, the

[7] Figures are numbered from left to right and top to bottom.

students would like to think in the language and concepts of dell'Abbaco's text. In Line 5, Maria talks about "slots". She says: "40 paces of the fox are 8 . . . 8 slots of the dog." In Line 8 she concludes that the dog travels 8 x 3 = 24 paces. The introduction of the term "slot" evokes a visual and kinesthetic experience that gives meaning to the multiplication 8 x 3. Still, the students feel that something is missing; Anna exclaims in Line 9, "I am turning crazy!" Maria eventually makes an important leap in the unfolding process of objectification (the process of noticing and understanding dell'Abbaco's mathematics) when she realizes that the fox is also moving forward (Line 12).

Despite being a problem on motion, the students' approach is based on the manipulation of signs. There is no space for sensuous activity such as gestures, manipulation of material objects, bodily movements, and rhythm. In Line 11, the teacher suggests a dynamic enactment of the moving bodies. In doing so, the teacher brings material objects and movement into the activity and *expresses*, in a different language—the language of gestures and actions with artifacts—the students' intuitions. In Line 12 Maria replicates the enactment of the motion of the moving bodies with her own gestures, as if rephrasing a 'gestural sentence' that has just been said. The recourse to gestures offers her the attainment of a new layer of understanding, of becoming conscious of the mathematical relations of the problem, realizing thereby that, in their calculations, the dog is chasing the fox but the fox does not move, and highlights the inconsistency—"but the fox does not stay still!" (Line 12). This emotional inconsistency that we see in these lines attests to the difference between Self and Other. There is still an 'objecting' *difference* between the students' understanding of the problem and dell'Abbaco's.

The students are aware that something is missing, that there is something that requires more articulation. Dell'Abbaco's mathematical way of thinking and speaking still remains elusive. Yet, in the previous dialogue we see the students making an effort to think and speak in a way that is foreign to them, as when we hear an unfamiliar language and try to formulate something in it with its own logic.

In terms of our theoretical framework, the episodes suggest that in their conversation with the past, through their gestures and utterances, the teacher and the students make their voices heard. The teacher's voice (Line 11) opens a space to imagine, in a different way, the motion under consideration. The students accept the teacher's invitation and engage in exploring the problem further. We see in these episodes the intertwined nature of the processes of objectification and subjectification. On the side of knowledge (objectification), to cope with the regular motion of the moving bodies, the students have come to the important idea of "slot". However, the question is not just about a formal mathematical understanding. Part of this conceptual mathematical understanding is an important affective dimension that is central to the processes of subjectification: in trying to find their voice while conversing with dell'Abbaco, part and parcel of a genuine understanding is, as the students show, being puzzled, confused, unsatisfied, curious. As we shall see in the following episodes, the process of subjectification (that started with the students' initial embarrassment) and the process of objectification (which encompasses the idea of "slots") will give rise to refinements in the encounter with cultural knowledge.

4.2 Seeing movement in space: Fixing the unit of measurement

Anna and Maria are now aware that the calculations should reflect the fact that the dog and the fox are both moving. They search for a suitable way of looking at the spatial relation between the pace of the dog and the pace of the fox.

13. Anna: Does it make sense to consider 1 pace of the dog and 1 pace of the fox? If we find the relation between the length of 1 pace of the dog with the length of the pace of the fox . . . ?
14. Teacher: Go ahead.
15. Maria: The pace of the dog is 5/3.
16. Anna: The pace of the dog is 5/3 of the pace of the fox *(They write the relation D = 5/3F).*

In Line 16, Anna and Maria have found a unit of measurement for space that corresponds to the length of the pace of the dog and, in Line 13, a conjecture about the relation between the movements of the fox and the movement of the dog. This is a subtle distinction that allows the introduction of pace as a marker of movement that we described in the analysis of dell'Abbaco's solution. The pace is like a time-marker and in each pace the dog advances 5/3F and the fox advances 1F *simultaneously*. However, at this point the students' pace as marker is conjecture only. This is why, despite their important achievement, the students are stuck. The correlation between the movement of the dog and the fox has not been accomplished yet.

Time as a physical quantity can be conceived of as a cultural-historical ideal form intertwined with movement itself and the artifacts we use. University textbooks of physics introduce time from the regularity of periodic motions provided by suitable artifacts that have evolved in what we call clocks. Albert Einstein, referring to Bridgman's operationalism (Bridgman, 1927), stresses the activity-bound meaning of time when he defines time as "what we measure with a clock" (Einstein in Gilder, 2009, p. 19). In his introduction to special relativity, Einstein describes the relationship between motion and time as follows:

> If we wish to describe the *motion* of a material point, we give the values of its co-ordinates as functions of the time. Now we must bear carefully in mind that a mathematical description of this kind has no physical meaning unless we are quite clear as to what we understand by "time." We have to take into account that all our judgments in which time plays a part are always judgments of *simultaneous events*. If, for instance, I say, "That train arrives here at 7 o'clock," I mean something like this: "The pointing of the small hand of my watch to 7 and the arrival of the train are simultaneous events." (Einstein, 1923, p. 86, emphasis in the original)

In this work, Einstein is questioning the foundations of Galilean relativity and Newtonian mechanics, in which motion had been formalized as a function of absolute time. In his dialogue with the past, Einstein is bringing a new voice as he encounters and transforms motion into an object of consciousness in perceptual and kinesthetic activity. The result is the dissension with, and subversion of, previous knowledge and the ensuing understanding of motion according to the tenets of special relativity. Einstein uses the small hand of the clock to measure movement, which is what we call time, in the same way as dell'Abbaco takes "pace" as a reasonable counter of motion. Anna and Maria, in their dialogue with the past, are confronted with the same shift from the clock as a measurement of absolute time to describe motion as a function of time, to the clock as a counter of motion. Anna and Maria at this point in the activity do not yet have any "hand of the watch" available to materialize this shift. The only measurable variable available in dell'Abbaco's problem is the length of the paces measured in a mutual relationship between the pace of the dog and the fox. From an abstract point of view, movement is not expressed in terms of a functional relationship between space and time but only in terms of the distance between the fox and the dog measured in paces and the cadence of paces.

Let us come back to Anna and Maria. Since there are no instances of time in its modern sense in dell'Abbaco's problem, Anna and Maria are struggling to find, within dell'Abbaco's way of thinking and speaking, a suitable expression for time. Time is *sensed* in the movement of the moving bodies but it is not *expressible* yet. There is still a gap between the pre-conceptual "*le dire"* (the saying) and the conceptual "*le dit"* (the said) (Lévinas, 1974). That is, there is a gap between that which one wants so much to say but still escapes the determining effect of utterances. To be *expressed*, time must acquire some determinations: it needs to become an object of discourse and consciousness (through, e.g., gestural, symbolic, numerical, and/or linguistic activity). The synchronic use of natural, embodied, and symbolic languages carves the space for sensuous and imaginative actions that, as we shall see in the following episodes, lead to a better understanding of the 14th century conception of motion.

4.3 The aesthetic of movement

After several attempts, Anna and Maria were not able to find the solution. The teacher prompted a *kinesthetic* reinterpretation of the problem.

17. Teacher: When the fox has walked his pace, which is a fox-pace long, the dog has walked his pace, which is 5/3 of the length of the fox-pace (*The teacher gestures the simultaneous movement of the fox and the dog, showing the length of the pace of the dog and the pace of the fox with the distance between each pair of fingers (Figure 2.1))*.
18. Anna and Maria: . . . which (the dog-pace) is longer.
19. Teacher: How can you go further?
20. Maria: When 5/3 becomes kind of an integer (*They are resorting again to calculation*).
21. Teacher: What is your objective?
22. Anna: (*understanding*) When the dog reaches the fox.
23. Maria: (*reformulating Anna's answer*) After how many paces the dog reaches the fox.
24. Teacher: (*The teacher shows the motion of the dog chasing the fox with his hands as in Figure 1.1*). What in the movement of my hands tells you that the dog is reaching the fox?
25. Anna: The dog goes faster. The space (the distance) between them decreases.
26. Teacher: How can you express the decrease of the distance in terms of paces? (*The students cannot answer. The teacher goes back to the gesturing as in Figure 2.1 showing how the gap between the animals decreases in each pace*).
27. Maria: This one (*distance between the animals*) continues decreasing because this one *(the dog)* continues going forward. So for each pace of the fox what is the decrease of the distance?
28. Anna: Correct, but how can I make it clear? (*The students laugh*)
29. Maria: (*Moving her hands in synchrony she enacts the decrease of the distance between the two animals (Figures 2.2 and 2.3)*). This one goes more forward (*the dog*) and this one (*the fox*) goes less forward; (*repeating*) this one (*the dog*) goes more forward and this one *(the fox)* goes less forward (*giving with words a time beat to her gesturing*) . . . they continue going forward . . . So, how much does the distance decrease? How much does the dog go more forward with respect to the fox in a pace? Always 2/3 (5/3 - 1) more. But! . . . It is not 2/3 with respect to the fox (*Maria finds it difficult to keep in mind both the difference between the step of the dog and the step of the fox (2/3 F) and the distance between the two animals (40 F) before they start moving).*
30. Teacher: What is the distance between the dog and the fox before they start moving?

31. Anna and Maria: 40 steps
32. Teacher: 40 steps of the fox. Then, after a pace, the fox is at 41 F and the dog is no more at 0 but at 5/3 F.

    After the calculation, Anna and Maria find that the new distance is 118/3 F and the difference from the previous one (40 - 118/3) is ⅔ F. They realize it is the same distance they calculated by subtracting the distance of the two steps independent of the initial distance between the two animals before movement starts.
33. Teacher: So, at every step the distance decreases by ⅔ F. How can I find in how many steps the dog catches the fox?
34. Maria: . . . So 2/3 . . . 40 steps . . . (*confident*). We have to see how many segments 2/3 long are in the 40 *(indicating the segment with her thumb and her index finger and gesturing in the air the segment covering at each pace the initial distance between the two animals; see Figures 2.4, 2.5, and 2.6).*
35. Anna: We have to divide 40 by 2/3

Anna and Maria understand that in 60 paces the dog catches the fox. They also perform a calculation to check their answer and they find that after 60 paces the dog and the fox are in the same position (60 x 5/3F = 40F + 60F). We remark that at this point Anna and Maria are confidently using "pace" as a counter of reappearing events; that is, their "little hand of the clock" we mentioned before.

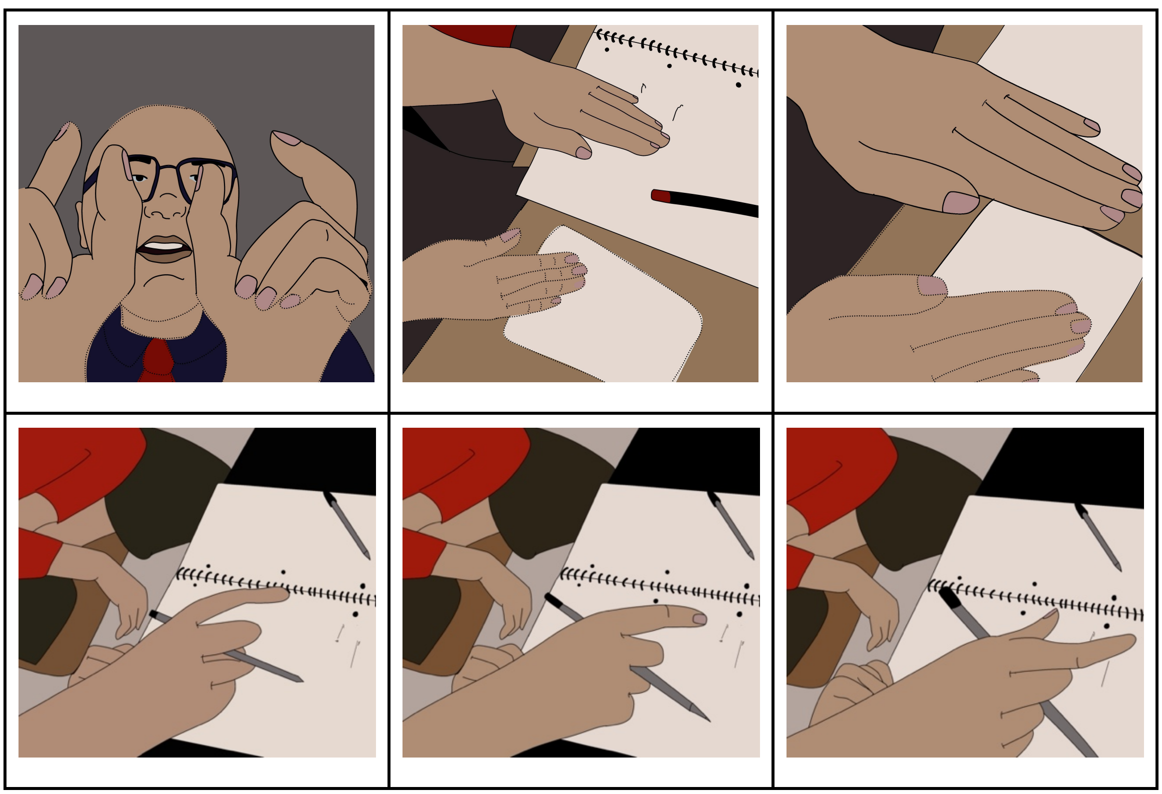

Fig. 2. In Figure 2.1, the teacher's gestures show the decrease of the original distance between the two animals as they make a pace. Maria's enactment of the counter of motion is shown, first qualitatively (Figures 2.2 and 2.3), then quantitatively (Figures 2.4–2.6).

The key element of this episode is imagining "pace" both as a measurement of space and as a counter of time. The elusive counter of motion manifests itself in Lines 26–29. In those lines, Maria finds her "small hand of the watch"; she finds a counter of movement while sensuously perceiving, touching, and feeling the *decrease* of the original distance between the two animals. She understands that at each pace (as a counter of time) the gap between the pace (as the length of the step) of the dog and the fox is 2/3 F but an important element is still missing for the solution of the problem. As illustrated by Maria's puzzlement at the end of Line 29, she does not yet link the 2/3 F gap corresponding to each time pace to the total decreasing distance between the two animals. She seems perceptively and kinaesthetically aware that at each pace such a distance decreases but she is struggling to quantify it. The teacher's voice allows the students to overcome this impasse. The solution to the problem was already present in Line 29 but it was not deemed meaningful in Maria's sensuous experience. Bit by bit, Anna and Maria come to perceive *quantitatively* that the decreasing distance between the animals was the outcome of their simultaneous march, each one with its specific length (5/3F for the dog and F for the fox). The "small hand of the watch" appears in the joint activity that intertwines Anna, Maria, and the teacher; it appears as something visual, tactile, aural, material, artifactual, gestural, kinesthetic, and rhythmical. Lines 21–35 are a paradigmatic example of voice as we understand it here; that is, as more than language and discursive activity.

The previous excerpts show a *process of objectification*. Through this process, the students came to find a way to think of and express key elements of dell'Abbaco's problem while remaining within the configuration of the historical arithmetic knowledge they are encountering. This encounter does not mean that their thinking has equalled dell'Abbaco's. This is not the point. "Understanding is not based on transposing oneself into another person . . . [it is] not to get inside another person and relive his experiences" (Gadamer, 1999, p. 383). The encounter with knowledge, as conceived in the theory of objectification, is the creation of a relational space where voices (past and present) enter into conversation; where individuals tune into each other trying to understand each other. The encounter with historical knowledge entails this effort of understanding dell'Abbaco that we see in the previous episodes.

Now, in the course of the process of objectification, the students come to position themselves vis-à-vis the Other's knowledge—through their deeds, their gestures, through what they say and think, and through what never comes into concrete existence and remains hinted at in the students' surprises, embarrassments, amazements, agonies, and joy.

Now, the unavoidably subjective experience that is intertwined with the encounter of knowledge is part of the students' *process of subjectification*, the process through which the students co-position themselves vis-à vis the Other. This positioning *changes* them as subjects of classroom mathematics practice. The changes implicated in the positioning, the acknowledgement of the Otherness of the Other, move them beyond the traps of synchronic knowledge, the present, and recognize in the Other (here dell'Abbaco and the mathematics system of knowledge his text embodies) the possibility to transcend the present and overcome what Lévinas (1974) calls *l'ipséité,* ipseity; that is, the self-centeredness of the 'I'.

4.4 Listening to dell'Abbaco's voice

The students spent some time reading and trying to understand dell'Abbaco's solution. They were particularly amazed by, and curious about, dell'Abbaco's use of 5 paces.

36. Maria: Concerning the solution of dell'Abbaco, it is strange reasoning with the arbitrary 5 paces and not in 1 pace. It is more complex. The way we conceive the solution is much more different.

We find here again the question of estrangement mentioned above. Maria's comment about dell'Abbaco's solution highlights the tension between past and present knowledge.

The teacher asked about the concept of time:

37. Teacher: What about the role of time in dell'Abbaco's solution?
38. Anna: I really don't see it. I could not say where the reference to time is.
39. Maria: It is identified with space itself. He [dell'Abbaco] doesn't calculate time but the number of paces required. Space and time are overlapped . . .
40. Teacher. As if time is in space itself!
41. Maria: Yes, time is in space, as if space has some form of regularity, a constant regularity . . . like time because the paces are always equal.
42. Anna and Maria: The paces are always equal. As if each pace is a unity . . .

In Lines 37–39 the students correctly recognize the implicit nature of time in dell'Abbaco's formulation and the solution of the problem; they argue that "space and time are overlapped." In Lines 41 and 42 they realize that "pace" is also a counter of motion. The regularity they refer to in Line 41 makes the counting of motion possible. In Line 42 "pace" appears as a counter of motion.

At the end of the two-session activity, Anna made the following remark:

43. Anna: The last time I was confronted with time-space problems was in high school, where you learn the formulae by heart, but you don't reason as we did here. I am happy I had the possibility to question the role of formulae as absolute dogma and understand their practical meaning.

These few lines highlight some of the differences noticed by the students. They also reveal important aspects of the students' *process of subjectification*, in particular their specific positioning vis-à-vis the Renaissance understanding of motion as they dialogue with Paolo dell'Abbaco.

5. Concluding remarks: The joint space of past and present presence

In this paper we attempted to address some aspects of Barbin, Guillemette, and Tzanakis's (2020) call about the need to reduce the gap between theoretical and practical investigations of history in mathematics education and the development of new specific themes of research. Against this background, in the first part of our paper, we presented two concepts that, often, are not explicitly thematized in the HPM research, namely, mathematics learning and classroom mathematics knowledge. We drew on the philosophy of dialectical materialism to present a concept of classroom mathematics knowledge as a *dynamic* and fluid *system*: this is a system of ways of mathematically thinking, reflecting, and doing that have been historically and culturally constituted. Mathematics learning, we suggested, can be theorized as the encounter with those ways of thinking, which occurs through intertwined processes of objectification and subjectification. The conception of learning that emerges from this dialectical materialist perspective led us to argue that the history of mathematics is not a mere tool to eventually enhance learning. The history of mathematics becomes a necessary component of the vital educational project of understanding our human nature as essentially historical and cultural; it allows us to transcend the limits of *ipseity* (the self-centeredness of the 'I'), which reduces mathematics knowledge to its present form.

When we claim that the history of mathematics is a necessity in mathematics education, we are not saying that students should be immersed in the problems of professional historians. Maybe we need to distinguish between History as a research field practiced by professional historians, and history (with lower case) as it appears in our claim, as the *ontological category* that is materialized in the world in an unfolding process of change; this is an ontological category that makes humans what we are, namely *historical* beings. Our claim is neither to 'use' the History nor the history of mathematics to encourage one of the many forms of otherness that can be imagined in a school context. Our claim is about *recognizing* history as an ontological category that works incessantly in the shaping of each one of us—in the ways we come to be and come to know. Of course, this recognition is intimately related to History (which, by the way, is also subjected to history as an ontological category). Hence, rather than denying the importance of History to mathematics educators, we consider History as a fundamental element in the imagination and creation of the pedagogical conditions that can help us to uncover what many (if not most) contemporary curricula have concealed behind an ahistorical presentation of mathematics and an ahistorical conception of teachers and students.

In the second part of the paper, we illustrated these ideas with an example in which prospective teachers entered into conversations with a 14th century mathematical problem from dell'Abbaco's *Trattato d'Aritmetica*. The key element of our approach is to be found in the idea of classroom teaching-learning *activity*. It is indeed through (and according to the didactic modalities of) classroom teaching-learning *activity* that the encounter with historical knowledge as embodied in dell'Abbaco's text is materialized. The historical mathematics knowledge is not something that reveals itself immediately to contemporary consciousness; it needs to be encountered and as such it is *mediated*. What mediates it is classroom teaching-learning *activity*.

In the hermeneutic tradition, the mediating element is not activity. It is *language*. "*Language is the universal medium in which understanding occurs*" (Gadamer, 1999, p. 389; emphasis in the original). It is through language that the students are seen as continuously interpreting and checking the plausible emerging interpretations. As Jahnke (2014) put it,

> You start with a certain image of the text reflecting your expectations about what it might be about. Then you read the text and realize that some aspects of your image do not agree with what is said in the source. Thus, you have to modify your image, read again, modify and so on until you are satisfied with the result or simply do not like to continue. (pp. 84–85)

Although hermeneutics has been one of the most productive frameworks in HPM research, and a very inspiring one, we prefer to see the students' engagement in the historical text in terms of *activity*. Classroom teaching-learning activity includes language, but it also includes a formidable range of non-discursive semiotic systems of expression, such as gesturing, kinesthetic actions, and instrument use, that, as we saw above, are crucial parts of the students' conversation with historical knowledge.

Through teaching-learning activity, historical mathematics knowledge presents itself as *different*. By being *different*, in its diachrony, the historical text challenges the synchrony of present knowledge. Classroom teaching-learning activity brings the diachronic and the synchronic into conversation. The question here is about accepting the challenge of the primal *difference* of

Otherness and the attempt to encounter the Other in its own Otherness as it is given to us in the text. The tremendous problem is to proceed to an encounter with past mathematics knowledge that does not assimilate it to our own conceptualizations. To assimilate it, to read it synchronically (see Fried, 2007), is to do it violence. Should we try to put on hold our own voices and conceptualizations? The problem with this line of thinking is that we cannot suppress ourselves from the act of understanding in order to receive and welcome the Other. Were we to put all our ideas and meanings on hold, we would hear nothing.

As the episodes from the Mathematics Laboratory suggest, the hearing of the Other—hearing dell'Abbaco talk about his problem—is *not* passive. The hearing is immediately reciprocated with *active* and *affective* understanding—with the students' own voices. The nature of the understanding of a historical text is to be *transformative*: in hearing the Other, what is heard is no longer the intact voice of history but the voice as the students grasp it, hence something new. We see that, from the outset, the encounter with history is not about a formal understanding of something written that conveys a definitive meaning set at once and forever.

By coming face to face with historical mathematics knowledge, the students create, through the teaching-learning activity they are producing with the teacher, a space that is neither theirs nor dell'Abbaco's; they create a *third* space, an in-between space, the space of the reception of historical mathematics knowledge. Let us call it the *joint space of past and present presence*.

This space is the space of shared time, where the past *challenges* the present and, by being challenged, the present *recognizes* itself in the past, as the saying may recognize itself in the said, without coinciding with it, as there will always be a remnant of the saying that can never be said and will never be said. In recognizing itself in the past, the present, however, resists absorbing the past. In the same way, the past resists absorbing the present (i.e., to see in the present its own image; to merely be '*un présent antérieur',* an earlier present). In the joint space, present and past acquire new meanings. Both appear to each other as something unfinished, as presents that never cease to be; they show to each other as presences that have never elapsed. They appear in relation, re-writing themselves mutually, transforming each other. In this joint space we understand "présence au passé et non présence du passé" (presence in the past and not presence of the past) (Dugravier, 2021, p. 41). In this line of thought, the present is understood

> not [as] the flattening of a diachrony which would be irreducible to it, but the place of the deployment of this diachrony; it does not level the differences off but can on the contrary, by summoning these differences, vivify them, put them in the present which is the only time of the shared life. (Dugravier, 2021, p. 40)

We see, then, that "the present is no longer what reduces diachrony to synchrony, but what bursts the latter into a time dilated between two poles whose center, moreover elusive and always shifting, constitutes the present" (Dugravier, p. 41).

Admittedly, we still need to explain the 'nature' and 'texture' or 'materiality' of the *joint space of past and present presence*. Its nature is to be polyphonic, full of tensions, diachronic and synchronic at the same time. Its texture is *relational*: it is made of the relations that we can establish with historical mathematical knowledge. But how? As Dugravier (2021) asks, "How to establish with the absolutely unknowable an authentic and respectful relationship with its subjectivity—and with mine?" (p. 20)

One possible answer would be in recognizing the modern truths in those of the past, considering truths as objective in the traditional sense, that is, *a*historical. This is an interesting view. However, it supposes a problematic substantialist view of truth, as if truth would always be the *same*, as if what cultures produce is the manifestation of the Spirit or the Mind that we mentioned when we referred to Hegel's work.

We would like to suggest that the genuine relationships that constitute the texture of the *joint space of past and present presence* are based on the acknowledgment of our differences and the responsible ethical recognition of the Other (in this case, historical mathematics knowledge). In the responsible ethical recognition of the Other, we acknowledge its presence on its own terms, discover its cultural-historical being and, by a kind of reflection, discover our own. Historical mathematics knowledge provides us with what we need to transcend our ipseity and to better understand ourselves. This is one of the priceless gifts the history of mathematics has to offer to mathematics education.

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